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Applications of the notion of analytic center in approximation (estimation) problems

György SONNEVEND *
Institut für Angewandte Mathematik, Universität Würzburg, Am Hubland, D-8700, Würzburg, FRG

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Abstract: A new method for solving a set of linear inequalities—based on a generalization of the central, "maximum entropy" solution of moment problems—is applied to problems of numerical approximation, cubature and filtering.

Keywords: Central solution of moment (inverse) problems, convex programming, filtering.

1. Introduction

In this paper we present several applications of a recently introduced concept, the analytical center solution of a moment problem, or what is the same, the analytical center of a polyhedron, see [8–10]. After reviewing the nice stability and invariance properties of this solution concept in the finite-dimensional case, giving hints for the computability of this solution by low complexity numerical algorithms, we concentrate on the following applications.

(1) Checking existence of multidimensional moment problems and solving extremal problems under moment-type conditions (generalized linear programming); solving problems of best uniform or L_1 approximation over multidimensional domains.

(2) Dynamic state estimation (smoothing, filtering) of input–output systems under uniformly bounded disturbance inputs and measurement errors.

(3) Constructing Gaussian-type quadrature formulae and stable Padé-type approximants ("minimal realizations").

(4) Recovering mass distributions (statistics, spectral estimation for stationary stochastic processes, recovering impedance functions of layered media.

Of course, due to limitations we can concentrate only on some aspects and only outline the new algorithms (based on the use of analytic centers and homotopies through them) for the solution of these problems, some of the other aspects of the above problems have also been dealt with in [6,8–14].

* On leave from Dept. of Numerical Analysis, Eötvös University, 1088 Budapest, Muzeum K.6-8, Hungary.

2. Basic properties of the analytic center of a polyhedron

For given vectors $k_1, \ldots, k_n \in \mathbb{R}^m$, scalars $c_1, \ldots, c_N \in \mathbb{R}^1$ and $m > N$, we denote by $P(k^N, c^N)$ the polyhedron, which will be supposed to have a nonempty (relative) interior,

$$P(k^N, c^N) := \{\mu \mid \langle k_i, \mu\rangle = c_i,\ i = 1, \ldots, N,\ \mu \geqslant 0\}, \tag{2.1}$$

i.e., P is the intersection of an $(m-N)$-dimensional linear manifold L with the positive orthant in $\mathbb{R}^m$. In the coordinates $x_1, \ldots, x_n$, $n = m - N$, of the affine space L this polyhedron can be equivalently represented as—using usual abbreviations, like $a^m = (a_1, \ldots, a_m)$—

$$P(a^m, b^m) = \{x \mid b_j \geqslant \langle a_j, x\rangle,\ j = 1, \ldots, m,\ x \in \mathbb{R}^n\}, \quad \text{where } a_1, \ldots, a_m \in \mathbb{R}^n, \tag{2.2}$$

the connection between these two descriptions being established by the relations $\mu_j = b_j - \langle a_j, x\rangle$, $j = 1, \ldots, m$. The analytic center of $P(k^N, c^N)$ is defined as the solution $\mu(k^N, c^N)$—which turns out to be unique, see [6]—of the maximum problem

$$\max\left\{\sum_{j=1}^{m} \log \mu_j \mid \mu \in P(k^N, c^N)\right\}. \tag{2.3}$$

The corresponding solution $x(a^m, b^m)$ in the x-space is defined by

$$\max\left\{\prod_{i=1}^{m} (b_i - \langle a_i, x\rangle) \mid x \in P(a^m, b^m)\right\}. \tag{2.4}$$

In [6] it is proved that the association $(a^m, b^m) \to x(a^m, b^m)$ is affine invariant (it is invariant also under scaling the side constraint functions) and it provides a two-sided ellipsoidal approximation of $P(a^m, b^m)$ around $x(a^m, b^m)$:

$$x(a^m, b^m) + \frac{1}{\sqrt{m-1}} E(a^m, b^m) \subseteq P(a^m, b^m) \subseteq x(a^m, b^m) + \sqrt{m-1}\, E(a^m, b^m), \tag{2.5}$$

where $E(a^m, b^m) := \{z \mid \langle D^2\psi(x(a^m))z, z\rangle \leqslant \psi(x(a^m, b^m))\}$;

$$\psi(x) := \left(\prod_{j=1}^{m} (b_j - \langle a_j, x\rangle)\right)^{1/m} \text{ is a concave function} \tag{2.6}$$

and D^2 denotes the matrix of second derivatives. A geometric characterization of $\mu(a^m, b^m)$ is that it is the barycenter of a simplex S in $\mathbb{R}^m$ such that

$$S \cap L = P(k^N, c^N). \tag{2.7}$$

The interior and exterior ellipsoids given above arise by intersection of L with the interior, resp. exterior ellipsoids standardly (affinely) associated to S. An important property of this solution concept easily generalized (for a system of convex, analytic inequalities, see below) is that the analytic center is a rather smooth, in fact, analytic function of the data (a^m, b^m) or (k^N, c^N). This is not shared by other solution concepts like the Tchebyshev center, the center of gravity or the center of ellipsoid of largest volume inside $P(a^m, b^m)$ (note that for a simplex, i.e., for $m = n + 1$, the analytic center is the barycenter). In fact $x(a^m, b^m)$ depends not only on $P(a^m, b^m)$ but also on those elements of (a^m, b^m) which perhaps do not "shape" the set $P(a^m, b^m)$. This is the price we have to pay for the smooth dependence on the data.

This will be exploited in computing this solution by smooth homotopies (path following) in which the parameters (a^m, b^m), resp. (k^N, c^N), in most cases only (one of) the values of b^m, resp. c^N are changed linearly in some parameter λ, and the corresponding solution $x(\lambda)$ or $\mu(\lambda)$ is followed by a predictor–corrector method. Since the variational equations for $x(a^m, b^m)$, resp. $\mu(k^N, c^N)$

$$\sum_{i=1}^{m} a_i(b_i - \langle a_i, x\rangle)^{-1} = 0. \tag{2.8}$$

resp.

$$K\mu = c^m, \qquad \alpha K = \mu^{-1}, \tag{2.9}$$

where $\alpha \in \mathbb{R}^N$ is the Lagrange multiplier, are rather simple (algebraically), the quadratically convergent Newton's method can be used as a corrector. Numerical experiments and details of implementation are presented in [6] (a theoretically more founded extrapolation, i.e., a prediction procedure is proposed in [9]) for the solution of linear programming problems (to which many researches applied recently these centers, see, e.g., [18])

$$\lambda^* = \min\{\langle c, x\rangle \mid x \in P(a^m, b^m)\}, \tag{2.10}$$

when one adds $\lambda \geqslant \langle c, x\rangle$, for $\lambda > \lambda^*$ to the conditions in $P(a^m, b^m)$ and continues the path of analytic centers $x(\lambda) = x(c, a^m, \lambda, b^m)$. A nice property of the method is that the above ellipsoidal approximations allow early detection of inactive constraints and the computation of asymptotically exact lower bounds, moreover the values $(b_j - \langle a_i, x\rangle)^{-1}(\lambda - \langle c, x\rangle)$, $i = 1, \ldots, m$ converge to the Lagrange multipliers for $\lambda \searrow \lambda^*$ (being for $\lambda > \lambda^*$ dual admissible).

3. Applications to semi-infinite problems of approximation

A natural generalization of the solution (2.4) to a feasibility problem: find x such that $f(x, s) < 0$, $s \in S$ where $f(x, s)$ is convex analytic in x, for each $s \in S$, and a measure $\mathrm{d}s$, defined on S, is the following

$$\sup \int_S \log(-f(x, s))\, \mathrm{d}s. \tag{3.1}$$

This is a concave analytic (i.e., nice) problem; the (generically) unique solution is an interior point of the feasible domain.

For example suppose that we have an approximation problem — in the uniform (or L_1) norm —over some domain S

$$\inf_{\alpha^N} \left\| f_0(s) - \sum_{i=1}^{N} \alpha_i f_i(s) \right\| =: \lambda^*, \tag{3.2}$$

where $f_0, \ldots, f_N$ are smooth, say analytic functions on S. We propose to use the homotopy path $(\epsilon(\lambda), \alpha^N(\lambda))$ defined by the solutions of the subproblems—defined for $\lambda > \lambda^*$—

$$\max\left(\log(\lambda - \epsilon) + \int_S \left(\log\left(\epsilon - f_0(s) + \sum \alpha_i f_i(s)\right) + \log\left(f_0 - \sum \alpha_i f_i(s) - \epsilon\right)\right) \mathrm{d}s\right), \tag{3.3}$$

resp.—for the L_1 norm, where $I(\eta) = \int_S \eta(\lambda, s)\, ds$—

$$\max\Big(\log(\lambda - I(\eta)) + \int_S \log\big(\eta(\lambda, s) - f_0 + \sum \alpha_i f_i(s)\big) + \log\big(f_0 - \sum \alpha_i f_i(s) - \eta(\lambda, s)\big)\, ds\Big). \tag{3.4}$$

Later we will compute the optimality conditions for similar problems (note that in (3.4) the unknowns are the functions $\alpha^N(\lambda)$ and $\eta(\lambda, s)$, $s \in S$, so the solution of (3.3) is much easier than that of (3.4).

In the dynamic state observation problem we have to recover (approximately) the state $x(t)$ of a dynamic system

$$\begin{aligned} \dot{x}(t) &= f(x(t)) + w(t), \quad \| w(t) \| \leqslant \rho, \\ \dot{y}(t) &= h(x(t)) + v(t), \quad \| v(t) \| \leqslant \sigma, \end{aligned} \tag{3.5}$$

based on the values of an error contaminated output, $y(s)$, $s \leqslant t$, when pointwise bounds ρ, resp. σ on the disturbance inputs and measurement errors are known. We propose to consider the variational problems—for each $T > 0$—

$$\sup \int_0^T \big(\log(\rho^2 - \| \dot{x} - f(x) \|^2) + \log(\sigma^2 - \| \dot{y} - h(x) \|^2)\big)\, dt, \tag{3.6}$$

where quadratic norms are assumed in (3.5) for w and v. In the case of a linear system: f and g linear in x, the problem is again a concave (i.e., nice) one with a unique solution $x(T, t)$ in general. This solution is characterized by the classical Euler–Lagrange equations together with the transversality conditions

$$\frac{d}{dt} x(T, t) = f(x(T, t)) \quad \text{for } t = 0 \text{ and } t = T.$$

Moreover one can derive an approximate dynamic observer

$$x' = g_0(x, z, y), \qquad z' = g_1(z, x, y),$$

where $x(t) = x(T, t)$ and $z(T)$ correspond to the adjoints of the transversality conditions at $t = 0$ transferred along the trajectory $x(T, \cdot)$ to $t = T$.

For brevity we omit the derivations; application for the discrete time problem (3.5) for linear systems has been analyzed in [14]. In the case of box constraints—say on w—we have to replace the first term in the integral by

$$\sum_{i=1}^{n} \big(\log(\rho_i - \dot{x}^i + f^i(x)) + \log(\dot{x}^i - f^i(x) - \rho_i)\big).$$

We consider now the infinite-dimensional analogon of the problem (and its solution) (2.1), (2.3). Let $K_i(s)$, $s \in S$, be continuous functions, $i = 0, 1, \ldots, N$ (again the special advantage of having smooth functions will be exploited by the easy computability of the integrals below over S in that case). The solution of the extremal problem under moment conditions: μ must be a nonnegative measure on S,

$$K_0^* = \inf\Big\{ \int_S K_0(s) \mu(ds) \mid \int_S K_i(s) \mu(ds) = c_i,\ i = 1, \ldots, N \Big\} \tag{3.7}$$

can be approached by computing the analytical centers $\mu(\lambda)$ for the set of measures satisfying the conditions in (3.3) and the additional condition

$$\lambda = \int_S K_0(s)\mu(\mathrm{d}s), \quad \lambda \geqslant K_0^*.$$

Postponing the existence questions for a moment, note that the rule of Lagrange multipliers, see (2.9), allows to characterize $\mu(\lambda)$ as

$$\mu'(\lambda, s) = \left(\sum_{i=0}^{N} p_i(\lambda) K_i(s) \right)^{-1}, \tag{3.8}$$

where $p^{N+1}(\lambda)$ is the solution of

$$\int_S K_i(s) \left(\sum_{i=0}^{N} p_i(\lambda) K_i(s) \right)^{-1} \mathrm{d}s = c_i, \quad i = 0, \ldots, N, \tag{3.9}$$

where $c_0 := \lambda$.

Here the left-hand side is —for each fixed λ—the gradient of a concave function of p^{N+1}. In order to decide whether the problem (3.7) has a feasible solution we propose the following homotopy approach. Let

$$\mu_0'(s) := \left(\sum_{i=1}^{N} p_i K_i(s) \right)^{-1} \tag{3.10}$$

be a positive, integrable (on S) function, whose moments are $\mathring{c}_i$, $i = 1, \ldots, N$, then obviously μ_0 is the analytic center of $K(K^N, \mathring{c}^N)$. Let

$$c^N(\lambda) = (1 - \lambda)\mathring{c}^N + \lambda c^N.$$

The set $K(K^N, c^N)$ is nonempty if and only if the path of analytic centers $\mu(\lambda)$ of $K(K^N, c^N(\lambda))$ exists for all $\lambda < 1$. Thus even if for $\lambda = \lambda^*$ the analytic center solution may not exist, while an atomic solution exists, the latter can be "approximated" arbitrarily well, in other words, the set c^N for which the analytic center solution exists, constitutes an open, dense subset of the (convex) set of all possible c^N.

Note that the analytic center solution is more smoothly depending on the moment data c^N (and K^N) than, e.g., the minimal atomic solutions (i.e., the analogons of Gaussian quadrature rules) do. In fact, the latter are analogons of extreme points of the feasible set (polyhedron) and as such they may change more abruptly under smooth (linear) homotopies of the data. Note that the analytic center remains continous even if the polyhedron changes not only its boundary topology but its dimension. It is a rather reassuring fact concerning the significance of this solution concept that in the classical Nevanlinna–Pick type moment problems, see [1,3–5,11,12], this solution, known as the "maximum entropy" solution, can be computed very easily (and recursively in N) in $O(N^2)$ arithmetical operations. The complexity in the finite-dimensional case is studied in [9] and [17].

The connections between maximum likelihood estimation and maximum entropy estimation have been studied by Akaike [2]. Not going into these topics further we would like to mention one immediate application to statistics, for the estimation of the density (distribution) function

of a random variable ξ. Suppose that the latter has the form

$$p(\theta, x) = \left(\sum_{i=1}^{N} \theta_i K_i(x) \right)^{-1},$$

where $\theta \in \mathbb{R}^N$ is an unknown parameter, and $K_1, \ldots, K_n$ are known functions. Suppose that $\xi_1, \ldots, \xi_M$ is a sequence of independent realizations of $\xi = \xi_\theta$. In order to estimate the underlying parameter θ we propose to compute the "empirical moments"

$$c_i := M^{-1} \sum_{j=1}^{M} K_i(\xi_j), \quad i = 1, \ldots, N,$$

and computing the estimate $\hat{\theta}$ by solving the equation

$$\int \left(\sum_j \theta_j K_j(s) \right)^{-1} K_i(s) = c_i, \quad i = 1, \ldots, N.$$

An application of this idea—together with a sequential selection of the functionals K^N—for the estimation of the spectrum of a stationary Gaussian process will be outlined in Section 5.

4. A method for computing cubature formulae

Even if we can estimate the value of an integral $\int K_0(s)\mu(\mathrm{d}s)$ based on moment information by solving two problems of the type (3.7), there is some interest to construct Gaussian cubature rules, i.e., minimal atomic solutions (μ^M, s^M) of moment problems

$$c_i = \sum_{k=1}^{M} \mu_k K_i(s_k), \quad i = 1, \ldots, N, \; s_i \in S, \tag{4.1}$$

with a possibly small value of M. The difficulty of this problem lies in its nonconvexity, e.g., even for the classical, polynomial moment problem on $\mathbb{R}^2$, where

$$K_i(s) = x^{i_1} y^{i_2}, \quad i = 1, \ldots, N, \; s = (x, y), \tag{4.2}$$

for natural index sets, like $0 \leqslant i_1 + i_2 \leqslant m$, the minimal solution is not unique in general. In the following approach we fix $M = N$, since, in a worst case sense (this is the "general" case!), one can construct, for the problem (4.2), domains S, such that, in the case of arbitrary m and of unit density over S, the minimal value of M is equal to N (the upper bound coming from Tsakalow's theorem, which is an application of a well-known theorem of Caratheodory, about representations of points of a convex body by convex combinations of extreme points); see, e.g., [7] for such an example.

In a first step we compute the analytic center solution μ, which is, in general, a smooth density. In the second step we solve the problem of near-best approximation of $\mu'(s)$ by piecewise constant functions $\gamma_N(s)$, i.e., those which are "measurable" with respect to a finite-dyadic subdivision of S (assumed to be a box $S = [a_1, b_1] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$, consisting of N elementary dyadic boxes over each of which γ_N is constant). In a third step we approximate the obtained piecewise constant density by an atomic density by simply concentrating the above densities in the centers of the corresponding boxes. It can be expected that the

atomic density so obtained can be used as a good approximation of a minimal atomic solution, i.e., as a starting point for Newton's method possibly in combination with a homotopy approach to solve the system (4.1), see below.

In order to compute the piecewise linear approximation of $\mu'(s)$, we use a sequential subdivision technique based on the principle of equalization of local errors. First of all we define the error of approximation of μ' by γ_N with the help of the "monotone" function of bounded "variation", $B_d(s)$ generated by a density $d(s)$ over S which is defined by the relation

$$\frac{\partial B_d(x)}{\partial x_1 \cdots \partial x_n} = d(x), \quad B_d(a_1, \ldots, a_n) = 0,$$

setting $\| \mu' - \gamma \| := \| B_{\mu'} - B_\gamma \|_{L^2(S)}$.

At a particular stage k of the subdivision algorithm we have a dyadic subdivision of $S = \bigcup_j S_j^k$ and select a particular subinterval S_{j*}^k to be subdivided next, such that S_{j*}^k will be replaced by 2^n new subintervals of equal volume to get the next subdivision of S. The selection of the value $J = J^*$ is accomplished by computing

$$\max_j \operatorname{vol}^{1/2}\left(S_j^k\right) \int_{S_j^k} \mu'(s)\, \mathrm{d}s. \tag{4.3}$$

After finishing the subdivision procedure, say—by an obvious misuse of notation—at stage N, the density γ_N is computed by setting

$$\gamma_N(s) = \operatorname{vol}^{-1}\left(S_j^N\right) \int_{S_j^N} \mu'(s)\, \mathrm{d}s \quad \text{for } s \in S_j^N. \tag{4.4}$$

One can prove, see [10], that the global, worst case error of this sequential recovery scheme is const $\cdot \sqrt{N}$ times smaller than that of any passive recovery scheme (i.e., of one, in which the subdivision is fixed, e.g., to be uniform). The moments corresponding to the density (4.4) will not be exactly equal to c^N, therefore we propose to compute for some $\epsilon > 0$, the analytic center of the inequality system

$$|c_i - \mathring{c}_i| \leqslant \epsilon, \quad i = 1, \ldots, N, \ \mathring{c}_i = \sum_{k=1}^{N} \mu_k K_i(x_k),$$

where x_k is the center of the box S_k^N, $k = 1, \ldots, N$. We can use the values of the atomic masses μ_k^N as starting points and set an appropriate value for ϵ. If $\epsilon = 0$ cannot be achieved we can use homotopies changing alternatively $\{\mu_k\}$ or $\{x_k\}$ for fixed $\{x_k\}$, resp. for a central value of c_i and fixed $\{\mu_k\}$ using Newton's method as a corrector. Here the use of "central" solutions assures an amount of freedom for changing $\{x_k\}$ when making ϵ smaller and smaller. An alternative method is to pursue the sequential subdivisions further to obtain an atomic solution with $\epsilon = 0$ (now having more than N atoms), which can be easily reduced to an atomic solution with at most N atoms by moving from the interior of the corresponding polyhedron to an extreme point of this polyhedron.

5. Applications to the modelling (estimation) of stationary processes

In previous papers [10–12] we proposed sequential methods for the approximation of positive real functions $\Omega(z)$ on the unit disc and for the recovery of the spectral mass $\mathrm{d}\mu$ associated to

elements of this class or what is the same: to a stationary, stochastic (Gaussian) process with covariance sequence $R_0, R_1, \ldots,$ where

$$\Omega(z) = R_0 + 2\sum_{k=1}^{\infty} R_k z^k, \quad |v| < 1, \quad R_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ik\theta}\, d\mu(\theta).$$

We demonstrated, that a suitable sequential choice of the nodes z_i, $i = 1, \ldots, N$,

$$z_{n+1} = A_{n+1}\big(z_1, \ldots, z_j, \Omega(z_1), \ldots, \Omega(z_j),\ j \leqslant n\big), \quad n \leqslant N,$$

allows a more accurate recovery of $\Omega = \Omega^{\mu}$ or μ in some interesting norms than the use of the passive information obtained by measuring the first N covariances, $R_0, \ldots, R_{N-1}$, see Theorem 1 and 2 in [11] and [10]. For μ we use the same norm, as above, i.e., the L_2 norm for the generated monotone functions, assuming only that the latter are continuous. Here we complement these results, where the final approximation for Ω—after N steps—is the maximum entropy estimation, i.e., an analytic center, realizing, see [5],

$$\max\left\{\int_{-\pi}^{\pi} \log \operatorname{Re}\, \Omega(i\theta)\, d\theta \,|\, \Omega(z_i) = c_i,\ i = 1, \ldots, N\right\}, \tag{5.1}$$

by some further remarks. First of all observe that given a finite-length realization of the underlying process the values of $\Omega(z_i)$, for $|z_i| < 1$ can be more easily estimated (especially for $|z_i|$ not too close to 1) than the (high-order) convariances. Indeed using the isometry of the map $x_k \to e^{-ik\theta}$ and the formulae for the elements of the Nevanlinna–Pick matrix

$$\sum \alpha_m \alpha_n \frac{\Omega(z_m) + \overline{\Omega(z_n)}}{1 - z_m \bar{z}_n} = \frac{1}{2\pi}\int \left|\sum_{m=1}^{N} \frac{\alpha_m}{e^{i\theta} - z_m}\right|^2 \mu(d\theta),$$

it is easy to provide "good" estimates for the value of

$$H_N = \frac{\Omega(z_m) + \overline{\Omega(z_n)}}{1 - z_m \bar{z}_n} \cong \left\langle \sum_{k=1}^{M} x_k z_m^k, \sum_{k=1}^{M} x_k z_n^k \right\rangle. \tag{5.2}$$

We assume x_k to be real, thus μ to be symmetric with respect to the real line. Of course the goodness depends on how large M is, how small $|z|$ is, anyway the asymptotic exactness is guaranteed by the ergodicity of the process (a consequence of the continuity of μ). We propose to continue the selection of new points, z_{n+1}, till the above matrix remains positive definite. Then we compute the maximum entropy solution $\Omega_N(z) = P_N(z)/q_N(z)$, e.g., by the Nevanlinna–Pick algorithm. In order to find a positive real, rational solution of lower degree for the same interpolation problem, we propose the following procedure. This is a generalization of a method to construct minimal, *stable* strictly proper (multipoint) approximations ("realizations") for transfer functions satisfying a set of interpolation conditions (e.g., having a set of fixed Markov parameters) presented in [14]. The question is to decide whether—for given $f < N$—there exists a positive-real transfer function $\Omega = p/q$, deg $p \leqslant$ deg $q \leqslant f$, satisfying a set of conditions

$$\Omega(z_i) = c_i, \quad |z_i| < 1, \quad i = 1, \ldots, N. \tag{5.3}$$

We propose first to "reduce" the order of the maximum entropy solution $\Omega(z^N, c^N)$ by some of the available stochastic model reduction procedures based on a singular value analysis, like the "canonical correlation analysis" or variants of Hankel norm optimal model (filter) reduction.

The reduced model of order f, $\mathring{p}/\mathring{q}$, in general, does not satisfy the interpolation conditions. Nevertheless we can expect that if there exist an fth order positive real interpolant p/q, then there also exist one, such that—under a normalization, like $q(0) = 1$—p and q are near to $\mathring{p}$ and $\mathring{q}$. In order to ascertain positive realness (and analyticity, i.e., zero-freeness in $|z| \leqslant 1$ for q) we impose the conditions—where $0 < \alpha < 1$ is an arbitrary constant—$|q(z) - \mathring{q}(z)|^2 \leqslant (1 - \alpha)|\mathring{q}(z)|^2$ for all $|z| \leqslant 1$ and the condition (nearly equivalent to the positivity of the real part of p/q): $s(z) > 0$, where $s(z) = 2\,\mathrm{Re}(\mathring{p} + \mathring{q})(\bar{p} + \bar{q}) - |p + q|^2$, for all $|z| = 1$. Using analytic centers, we propose to solve the problem

$$\max\left\{\int_D \log\left((1-\alpha)|\mathring{q}(z)|^2 - |\mathring{q}(z) - q(z)|^2\right) r\,\mathrm{d}r\,\mathrm{d}\theta + \int_T \log\left(s(\mathrm{e}^{\mathrm{i}\theta})\right)\mathrm{d}\theta\right\} \tag{5.4}$$

under the side conditions $p(z_i) = c_i q(z_i)$, $i = 1, \ldots, N$. Note that for $\mathring{c}_i = \mathring{p}(z_i/\mathring{q}(z_i)$, $i = 1, \ldots, N$, the analytic center is just $\mathring{p}(z)/\mathring{q}(z)$, we can use homotopy $c^N(\lambda) = (1 - \lambda)\mathring{c}^N + \lambda c^N$ for following the path of coefficients of $p(\cdot)$ and $q(\cdot)$; the integrals arising over the disc D and the circle T are concave analytic functions of these coefficients, the side conditions being linear in them. Above we have chosen the linearization (around (p, q)) of the "nonconvex part" in the exact condition for positive realness: $|p + q| > |p - q|$. The complete solution of the above problem is known to be very difficult, see [3,13,16]. Finally note that a regularization procedure taking into account the errors in the measurement of the values of $\Omega(z_i)$, $i = 1, \ldots, N$, (in fact the latters should be normalized with a factor $(1 - |z_i|^2)^{-1}$) could be following: solve the concave analytic problem, where $\gamma > 0$ is a fixed parameter,

$$\max\left\{\det\left(\frac{c_i + \bar{c}_j}{1 - z_i\bar{z}_j}\right) + \gamma\sum_{i=1}^{N}\log\left(\epsilon_i - (c_i - \tilde{c}_i)\right) + \log\left(c_i - \tilde{c}_i - \epsilon_i\right)\right\}, \tag{5.5}$$

where ϵ_i, $i = 1, \ldots, N$, stand for the upper bounds of errors of estimation.

If the solution $\mathrm{d}\mu$ is sought for among the atomic ones with a minimal number of atoms (estimation of periodicities, i.e., "sinusoids in white noise"), we have to find the smallest eigenvalue of the above matrix and after that solve a full eigenvalue problem for a unitary matrix as explained in [11], see also [4]. The use of sequential node selection allows to concentrate z^N around these mass concentration which is an obvious advantage. Note that the solution of (5.5) is an analytic center. Letting γ converge to zero we obtain a further interesting solution. In fact, the solution of the problem (5.1)—in the case of exact interpolation data—maximizes the determinants $H_{\hat{N}}$ for all $\hat{N} > N$ simultaneously, i.e., when we assume that only those elements of $H_{\hat{N}}$ are fixed, which belong to H_N.

Similar problems of optimal extension for partially defined matrixes can also be related to analytic centers. For example, the problem of minimum norm extension (completion)—say, in the class of symmetric matrices—

$$\min\left\{\|B\| \,|\, B_{ij} = A_{ij},\ (i, j) \in J,\ B = B^*\right\} =: \mu^* \tag{5.6}$$

can be approached, using homotopy in $\mu > \mu^*$, by computation of the "central extensions", i.e., solutions of

$$\max\left\{\det(\mu I - B)\det(B + \mu I) \,|\, B_{ij} = A_{ij},\ (i, j) \in J,\ B = B^*\right\}.$$

Other interesting extensions arise when we use, instead of the determinant function, other "penalty" functions, e.g.,

$$\max\left\{\int_G \log\langle Bb,\ b\rangle\ \mathrm{d}b \mid B_{ij} = A_{ij},\ (i,\ j) \in J,\ B = B^* \geqslant 0\right\}, \tag{5.7}$$

$$\max\left\{\int_G \log\ \mu'(b)\ \mathrm{d}b \mid B_{ij} = \int \mu(b) b_i b_j\ \mathrm{d}b = a_{ij},\ (i,\ j) \in J,\ \mu' \geqslant 0\right\}; \tag{5.8}$$

here G denotes the unit ball in $\mathbb{R}^m$. Similar problems, like (5.6)–(5.8) play an important role in the theory of operators ("dilatation theory", see [15]).

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